

# A Course on Noncommutative Geometry in String Theory

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In this pedagogical mini course the basics of the derivation of the noncommutative structures appearing in string theory are reviewed. First we discuss the well established appearance of the noncommutative Moyal-Weyl star-product in the correlation functions of open string vertex operators on a magnetized D-brane. Second, we will review the most recent attempts to generalize these concepts to the closed string moving in a nongeometric flux background.

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## 1 Introduction

It is well known that Einstein's theory of gravity, i.e. general relativity, cannot be consistently quantized with the rules developed in the framework of quantum field theory. Using perturbation theory around a flat metric, the ultraviolet divergences require infinitely many counter-terms.

Thus, it is natural to consider theories in which a natural minimal length is imposed. With the lessons from quantum theory, where a minimal volume element in phase space is introduced, one could imagine introducing noncommuting coordinates to generate a minimal volume element in space-time. This is the idea of noncommutative geometry, which by now is mathematically well developed and whose physical applications is the subject of this workshop. What is still lacking is a concrete idea how to use this formalism to formulate a consistent theory of quantum gravity. In fact noncommutative geometry has mainly been employed for applications to generalized gauge theories, like the noncommutative standard model of Connes-Lott [1].

Another approach to introduce a minimal length is by postulating that the fundamental physical objects are extended so that their interaction also violates the principle of locality. Certainly, the most prominent example is string theory, for which the existence of gravity is a direct consequence and which contains infinitely many new ultraviolet degrees of freedom, namely all the higher oscillation modes of the string, making the string theory on-shell Feynman diagrams finite.

The question is whether the above mentioned two approaches are related or maybe even complementary to each other. Since string theory is by far better developed, one can approach this question by looking for the appearance of noncommutative target-space structures in both open and closed string theory. For the open string there exists a well established relation, namely open strings ending on a D-brane supporting a non-vanishing two-form (gauge flux) see a noncommutative D-brane world-volume geometry. In fact, as shown by Seiberg-Witten [2] one can describe in a certain limit the effective theory on the brane as a noncommutative gauge theory. Then, the star-product in Witten's open string field theory [3] is directly related to the Moyal-Weyl star-product for functions on the target space.

During the recent years there have been some attempts to reveal similar noncommutative structures in closed string theory [4–13]. As will be explained in this course, in this case the analysis is far more subtle and seems to be obscured by the limitation of only having available a background dependent formulation of string theory. That means that the background satisfies the string equations of motion, which by definition are the constraints for conformal invariance of the sigma model action. There are certain indications that the closed string is related to the appearance of not only noncommutative but even nonassociative target

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\* Lecture notes for a 3h course given at the Workshop on Noncommutative Field Theory and Gravity September 8 - 15, 2013, Corfu

space geometries. It is fair to say that the precise relation is still not completely settled, but a couple of interesting observation have been reported.

In this mini-course a pedestrian introduction into the appearance of noncommutative geometry in both the open and the closed string sector is provided. It is aimed not for specialists in string theory but rather for people interested in noncommutative geometry. This is why the course starts at a considerable basic level. Moreover, a partial overlap with the already existing proceeding article [14] could not be avoided.

## 2 String theory preliminaries

In this section, let us introduce some basic material of the bosonic closed and open string. We will restrict ourselves to those aspect which will become relevant in the course of this lecture. For more details and derivations we refer the reader to the existing text books [15–17].

### Aspects of closed string world-sheet theory

The usual approach to string theory is in a first quantized version, i.e. one considers a string moving in a target space background with metric  $g_{ab}$ , Kalb-Ramond field  $B_{ab}$  and dilaton  $\Phi$ , whose dynamics is governed by a two-dimensional non-linear sigma model. With  $\Sigma$  denoting the world-sheet of the closed string, its action reads

$$\mathcal{S}_{\text{bulk}} = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\sigma d\tau \left( h^{ij} g_{ab}(X) \partial_i X^a \partial_j X^b + \varepsilon^{ij} B_{ab}(X) \partial_i X^a \partial_j X^b \right), \quad (1)$$

where  $h^{ij}$  denotes the Minkowski world-sheet metric and we suppressed the dilaton part. Here the coordinate fields depend on the world-sheet coordinates  $(\sigma, \tau)$  and satisfy the boundary condition  $X^a(\sigma + 2\pi, \tau) = X^a(\sigma, \tau)$  for closed strings. Choosing for instance the simple background of a flat metric  $g_{ab} = \eta_{ab}$  and  $B_{ab} = 0$ , the 2D equation for motion is simply the wave equation  $(\partial_{\tau}^2 - \partial_{\sigma}^2)X^a = 0$ . Introducing light-cone coordinates  $u = \tau + \sigma$ ,  $v = \tau - \sigma$ , this becomes  $\partial_u \partial_v X^a = 0$ . Its solution splits into left and right moving waves  $X^a(\sigma, \tau) = X_L^a(u) + X_R^a(v)$  with the mode expansion

$$\begin{aligned} X_L^a(u) &= \frac{x_0^a}{2} + \frac{\alpha'}{2} P^a(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n}{n} e^{-in(\tau + \sigma)}, \\ X_R^a(v) &= \frac{x_0^a}{2} + \frac{\alpha'}{2} P^a(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\bar{\alpha}_n}{n} e^{-in(\tau - \sigma)} \end{aligned} \quad (2)$$

where  $x_0^a$  denotes the center of mass position and  $P^a$  the center of mass momentum operator of the closed string. One also defines  $\alpha_0^a = \bar{\alpha}_0^a = \sqrt{\frac{\alpha'}{2}} P^a$ . The modes satisfy the commutation relations

$$[\alpha_m^a, \alpha_n^b] = \eta^{ab} m \delta_{m, -n}, \quad [\bar{\alpha}_m^a, \bar{\alpha}_n^b] = \eta^{ab} m \delta_{m, -n}, \quad [\alpha_m^a, \bar{\alpha}_n^b] = 0. \quad (3)$$

Thus, one can define the modes  $\alpha_n^a$  and  $\bar{\alpha}_n^a$  with  $n < 0$  as raising operators and the positive modes as lowering operators. The Hilbert space is then given by the Fock-space of the two algebras acting on a ground state  $|\vec{p}\rangle$  with  $P^a|\vec{p}\rangle = p^a|\vec{p}\rangle$  subject to the level matching constraint  $N_L - N_R = 0$  with  $N_{L/R}$  denoting the left and right moving occupation numbers. The target space mass of such a state is given by  $\frac{\alpha'}{2} M^2 = N_L + N_R - 2$ .

For utilizing the powerful methods of complex analysis, one Wick rotates the world-sheet into Euclidean signature and defines  $z = \exp(\tau + i\sigma)$ . From the commutator algebra one can compute the two-point function as

$$\langle X^a(z, \bar{z}) X^b(w, \bar{w}) \rangle = -\alpha' \eta^{ab} \log |z - w|. \quad (4)$$

For each state in the Hilbert space  $|\phi\rangle$ , there exist a corresponding field  $\phi(z, \bar{z})$ , where the correspondence is determined via  $\lim_{z \rightarrow 0} \phi(z)|0\rangle = |\phi\rangle$ . For instance, the ground state tachyon of mass  $\frac{\alpha'}{2}M^2 = -2$  and momentum  $\vec{p}$  corresponds to the field (vertex operator)  $V_T = \exp(i\vec{p} \cdot \vec{X}(z, \bar{z}))$  with  $\frac{\alpha'}{2}\vec{p}^2 = 2$ . At the first excited level one finds the on-shell vertex operator

$$V_G = \xi_{ab} \partial X^a(z) \bar{\partial} X^b(\bar{z}) \exp(i\vec{p} \cdot \vec{X}(z, \bar{z})) \quad (5)$$

with  $\vec{p}^2 = 0$  and transverse polarization, i.e.  $p^a \xi_{ab} = p^b \xi_{ab} = 0$ . For symmetric polarization this is the graviton/dilaton and for anti-symmetric one the Kalb-Ramond field. The tree level closed string scattering amplitude for  $N$  such on-shell vertex operators is defined as

$$\mathcal{A} \sim \int d^2 z_1 \dots d^2 z_N \prod_{i=1}^3 \delta(z_i - z_i^0) |z_{12} z_{13} z_{23}|^2 \langle V_1(z_1, \bar{z}_1) \dots V_N(z_N, \bar{z}_N) \rangle_{S^2} \quad (6)$$

where we included the universal contribution from the ghosts and already implemented the effect of the  $SL(2, \mathbb{C})$  invariance, which allows to move the first three points to three fixed ones  $z_i^0$ . The most common choice is  $z_1 = \infty, z_2 = 1, z_3 = 0$ . The correlation function below the integral is to be computed in the conformal field theory on the Riemann sphere (compactified complex plane). Higher loop amplitudes involve CFT correlation functions on higher genus Riemann surfaces.

Note that in the mode expansion (2) the zero mode part is left-right symmetric. This changes when one considers the compactification of the string on e.g. a circle of radius  $R$  in the  $a = 25$  direction. In this case, starting with the general ansatz  $X_{L/R}^{25}(u) = \frac{x_0^{25}}{2} + \frac{\alpha'}{2} p_{L/R}^{25}(\tau \pm \sigma) + \dots$ , one has quantized Kaluza-Klein momentum  $(p_L^{25} + p_R^{25})/2 = m/R$ ,  $m \in \mathbb{Z}$  and quantized winding strings  $X^{25}(\tau, 2\pi) = X^{25}(\tau, 0) + 2\pi n R$ ,  $n \in \mathbb{Z}$ . Thus, one gets

$$p_{L/R}^{25} = \frac{m}{R} \pm \frac{nR}{\alpha'} \quad \text{with mass spectrum} \quad \alpha' M^2 = \alpha' \frac{m^2}{R^2} + \frac{1}{\alpha'} n^2 R^2 + \dots \quad (7)$$

This is invariant under the transformation

$$T : R \rightarrow \frac{\alpha'}{R}, \quad m \leftrightarrow n. \quad (8)$$

This acts on the left and right momenta as  $T : (p_L^{25}, p_R^{25}) \rightarrow (p_L^{25}, -p_R^{25})$  and can be extended to the entire string theory. Thus, closed strings on a circle admit a new type of symmetry, so called T-duality, which acts on the coordinates in a left-right asymmetric way  $T : (X_L^{25}, X_R^{25}) \rightarrow (X_L^{25}, -X_R^{25})$ .

### Target space equations of motion

From the computation of string scattering amplitudes one can deduce an effective target space action for the massless modes, i.e. the graviton, the dilaton and the Kalb-Ramond field. Alternatively, one can start from the non-linear sigma model and require that on-shell backgrounds are given by its conformal fixed points. From the 2D point of view, the coupling constants are given by the target space background fields. This quantum field theory is treated perturbatively in a dimensionless coupling  $\sqrt{\alpha'}/R$ , where  $R$  is a characteristic length scale of the background. At leading order, the beta-function equations for the couplings  $g_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$  read

$$\begin{aligned} 0 &= \beta_{ab}^G = \alpha' \left( R_{ab} - \frac{1}{4} H_a{}^{cd} H_{bcd} + 2 \nabla_a \nabla_b \Phi \right) + O(\alpha'^2), \\ 0 &= \beta_{ab}^B = \alpha' \left( -\frac{1}{2} \nabla_c H^c{}_{ab} + \alpha' H_{ab}{}^c \nabla_c \Phi \right) + O(\alpha'^2), \\ 0 &= \beta_{ab}^\Phi = \frac{1}{4} (d - d_{\text{crit}}) + \alpha' \left( (\nabla \Phi)^2 - \frac{1}{2} \nabla^2 \Phi - \frac{1}{24} H^2 \right) + O(\alpha'^2). \end{aligned} \quad (9)$$

The first equation, at leading order in  $\alpha'$ , is nothing else than Einstein's equation with sources. Clearly, in this approach one is assuming from the very beginning that the string is moving through a Riemannian geometry with additional smooth fields. However, it is well known that there exist conformal field theories which cannot be identified with such simple geometries. These are left-right asymmetric like for instance asymmetric orbifolds. The latter are asymmetric at some orbifold fixed points but one can imagine asymmetric CFTs which are not even locally geometric. We will see that the target space interpretation of such asymmetric CFTs is closely related to noncommutative geometry.

### Aspects of open strings

Now consider open strings with  $0 \leq \sigma \leq \pi$ . Such open strings can end on D-branes carrying in general a non-vanishing gauge flux  $F_{ab} = \partial_a A_b - \partial_b A_a$  on their world-volume. One extends the non-linear world-sheet sigma-model action (1) to include the gauge fields on the D-branes at the end of the open string world-sheet

$$S = \mathcal{S}_{\text{bulk}} - \int_{\partial\Sigma} d\tau A_a(X) \partial_\tau X^a. \quad (10)$$

The action has the Abelian gauge invariance of the vector potential at the boundary  $\delta A_a = \partial_a \Lambda$  and the combined two-form gauge invariance of the antisymmetric tensor  $B_{ab}$ , which also involves a boundary term,

$$\delta B_{ab} = \partial_a \zeta_b - \partial_b \zeta_a, \quad \delta A_a = -\frac{1}{2\pi\alpha'} \zeta_a. \quad (11)$$

For constant metric  $G_{ab}$ , antisymmetric tensor  $B_{ab}$  and gauge field strength  $F_{ab}$  the action (10) simplifies. First we notice that, using

$$\int_{\Sigma} d^2\sigma \varepsilon^{ij} B_{ab} \partial_i X^a \partial_j X^b = \int_{\partial\Sigma} d\tau B_{ab} X^a \partial_\tau X^b, \quad (12)$$

the term involving  $B_{ab}$  can be written as a boundary term. For constant  $F_{ab}$  one can also write

$$\int_{\partial\Sigma} d\tau A_a \partial_\tau X^a = \frac{1}{2} \int_{\partial\Sigma} d\tau F_{ab} X^a \partial_\tau X^b \quad (13)$$

so that the total world-sheet action becomes

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\sigma d\tau \eta^{ij} g_{ab} \partial_i X^a \partial_j X^b - \frac{1}{4\pi\alpha'} \int_{\partial\Sigma} d\tau (B_{ab} + (2\pi\alpha') F_{ab}) X^a \partial_\tau X^b. \quad (14)$$

Therefore, the end points of the open string couple to the gauge invariant field strength

$$2\pi\alpha' \mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab}. \quad (15)$$

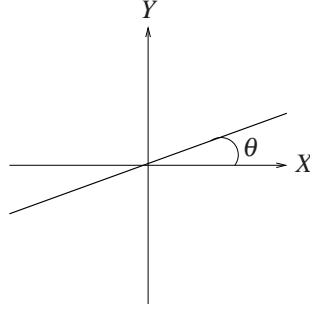
The boundary conditions at  $\sigma = 0, \pi$  that follow from the variation of the world-sheet action are

$$g_{ab} \partial_\sigma X^b + (2\pi\alpha' \mathcal{F}_{ab}) \partial_\tau X^b|_{\sigma=0,\pi} = 0. \quad (16)$$

For the target space  $T^2$  with flat metric  $g_{ab} = \delta_{ab}$  and the constant field strength  $\mathcal{F} = \mathcal{F}_{12}$ , we obtain the so-called mixed boundary conditions

$$\begin{aligned} \partial_\sigma X + (2\pi\alpha' \mathcal{F}) \partial_\tau X &= 0, \\ \partial_\sigma Y - (2\pi\alpha' \mathcal{F}) \partial_\tau X &= 0. \end{aligned} \quad (17)$$

Thus, the gauge field interpolates between a Neumann boundary condition  $\partial_\sigma X^a = 0$  and a Dirichlet boundary condition  $\partial_\tau X^a$ .



**Fig. 1** D-brane at angle

For an open string with either pure Neumann or pure Dirichlet boundary conditions on both ends, one obtains the following mode expansion

$$X^a(z)_{\text{NN/DD}} = x_0^a + 2\alpha' P^a \log|z| + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} (z^{-n} \pm \bar{z}^{-n}) \quad \text{with } [\alpha_m^a, \alpha_n^b] = \eta^{ab} m \delta_{m,-n}. \quad (18)$$

Thus, relative to the closed string the degrees of freedom are halved. The open string two-point function for these two cases can be straightforwardly computed as

$$\langle X^a(z) X^b(z') \rangle_{\text{NN/DD}} = -\alpha' \delta^{ab} (\log|z - z'| \pm \log|z - \bar{z}'|) \quad (19)$$

where  $z$  and  $z'$  take values in the complex upper half plane. Now we discuss how T-duality acts in the open string sector. Since it exchanges momentum and winding, a T-duality in the  $X$ -direction acts as  $T : \partial_\tau X \leftrightarrow \partial_\sigma X$ . Let us consider a D-brane on non-compact  $\mathbb{C}$ , which is shown in figure 1. The D-brane intersecting the  $X$ -axis under an angle  $\theta$  provides the following boundary condition for an open string

$$\begin{aligned} \cos \theta \partial_\sigma X + \sin \theta \partial_\sigma Y &= 0, \\ -\sin \theta \partial_\tau X + \cos \theta \partial_\tau Y &= 0. \end{aligned} \quad (20)$$

Applying a T-duality in the  $Y$ -direction, these boundary conditions become

$$\begin{aligned} \partial_\sigma X + \tan \theta \partial_\tau Y &= 0, \\ \partial_\sigma Y - \tan \theta \partial_\tau X &= 0. \end{aligned} \quad (21)$$

These are precisely of the mixed type (17) so that we can identify  $(2\pi\alpha' \mathcal{F}) = \tan \theta$ . Under T-duality in the  $Y$ -direction a  $D1$ -brane at angle is mapped to a magnetized  $D2$ -brane. Moreover, a (left-right symmetric) rotation of the  $D1$ -brane by an angle  $\theta$

$$\vec{X}_L \rightarrow A \vec{X}_L, \quad \vec{X}_R \rightarrow A \vec{X}_R, \quad \text{with} \quad A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (22)$$

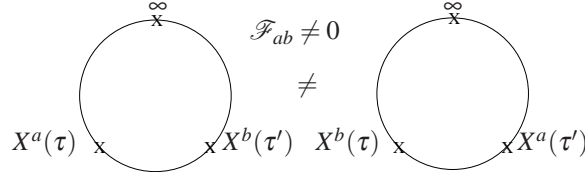
is mapped under T-duality to a left-right asymmetric rotation defined as

$$\vec{X}_L \rightarrow A \vec{X}_L, \quad \vec{X}_R \rightarrow A^T \vec{X}_R. \quad (23)$$

Thus we summarize that left-right asymmetric symmetries of the world-sheet act in the open string sector via changing the magnetic flux on the brane.

### 3 Noncommutative geometry: open strings

That noncommutative geometry is related to open string theory has long been suspected. For instance, in Witten's cubic open string field [3] the fundamental product between two string fields is noncommutative,



**Fig. 2** Disc diagram

though still associative and cyclic. Expanding the theory around the flat background with vanishing Kalb-Ramond field, this noncommutativity is not visible in the zero mode sector, i.e. the target space coordinates are still commutative. However, by turning on a background which is sensitive to the order of two end-points of an open string, one has a chance to also make manifest the noncommutativity in the zero mode sector. As we will derive in this section, such a background is given by a constant  $\mathcal{F}_{ab}$  flux. Mathematically this means that for non-vanishing  $\mathcal{F}_{ab}$  the two disc correlators shown in figure 2 are different. In the following we will compute this amplitude using the methods presented in the previous section and which were introduced in [18].

We are still working on a two-dimensional flat target space and recall the two-point function (19) on the upper half plane for NN boundary conditions

$$\begin{aligned} \langle X^a(z) X^b(z') \rangle &= -\alpha' \delta^{ab} (\log |z - z'| + \log |z - \bar{z}'|) \\ &= -\alpha' \delta^{ab} \frac{1}{2} (\log(z - z') + \log(\bar{z} - \bar{z}') + \log(z - \bar{z}') + \log(\bar{z} - z')) . \end{aligned} \quad (24)$$

In the second line we have split the two-point function so that, formally using

$$X^a(z) = X_L^a(z) + X_R^a(\bar{z}), \quad (25)$$

we can directly read off the individual contributions from the left- and right-movers. According to our previous discussion, we can turn on a non-vanishing constant flux by performing an asymmetric rotation (23). A straightforward computation leads to the following expression for the propagator in the rotated coordinates

$$\begin{aligned} \langle X^a(z) X^b(z') \rangle &= -\alpha' \delta^{ab} \log |z - z'| - \alpha' \delta^{ab} (\sin^2 \theta - \cos^2 \theta) \log |z - \bar{z}'| \\ &\quad - \alpha' \epsilon^{ab} \sin \theta \cos \theta \log \left( \frac{z - \bar{z}'}{\bar{z} - z'} \right) . \end{aligned} \quad (26)$$

Using  $\tan \theta = B_{12} = B$  we can write this as

$$\begin{aligned} \langle X^a(z) X^b(z') \rangle &= -\alpha' \delta^{ab} (\log |z - z'| - \log |z - \bar{z}'|) - \frac{2\alpha'}{1+B^2} \delta^{ab} \log |z - \bar{z}'| \\ &\quad - \frac{\alpha' B}{1+B^2} \epsilon^{ab} \log \left( \frac{z - \bar{z}'}{\bar{z} - z'} \right) . \end{aligned} \quad (27)$$

Finally, restricting this to the real axis,  $z = \bar{z} = t$  and  $z' = \bar{z}' = t'$ , and choosing the branch cut of the logarithm in the second line of (27) along the negatively imaginary axis one obtains

$$\langle X^a(t) X^b(t') \rangle = -\frac{\alpha'}{1+B^2} \delta^{ab} \log(t - t')^2 + \frac{i\pi\alpha'B}{1+B^2} \epsilon^{ab} \epsilon(t - t') . \quad (28)$$

Here  $\varepsilon(x) = 1$  for  $x$  positive and  $\varepsilon(x) = -1$  for  $x$  negative. Note that the second term is not symmetric under exchange of  $X^a$  and  $X^b$  and therefore is the source of noncommutativity. This expression agrees precisely with the propagator derived in [2] with the identification of the open string metric

$$G^{ab} = \frac{1}{1+B^2} \delta^{ab} \quad \text{and} \quad \theta^{ab} = 2\pi\alpha' \frac{B}{1+B^2} \varepsilon^{ab}. \quad (29)$$

Thus, by applying an asymmetric rotation we have found an elegant and short way of deriving this propagator. This relation of the open and closed string quantities can be compactly written as

$$\left( G^{-1} + \frac{1}{2\pi\alpha'} \theta \right) = (g + 2\pi\alpha' \mathcal{F})^{-1}. \quad (30)$$

Using the correlator (28), one can compute the operator product expansion of open string tachyon vertex operators at the boundary

$$e^{ipX}(t) e^{iqX}(t') = (t-t')^{2\alpha' G^{ab} p_a q_b} \exp\left(-\frac{i}{2} \theta^{ab} p_a q_b\right) e^{i(p+q)X}(t') + \dots \quad \text{for } t > t'. \quad (31)$$

The extra phase is independent of the world-sheet coordinates and can therefore be considered as a target-space effect. Indeed, this phase can be considered to arise due to a noncommutative deformation of the product of functions on the target space, namely from the Moyal-Weyl star-product

$$(f \star g)(x) = \exp\left(\frac{i}{2} \theta^{ab} \partial_a^{x_1} \partial_b^{x_2}\right) f(x_1) g(x_2) \Big|_x. \quad (32)$$

This noncommutative product satisfies a couple of properties:

1. For the coordinate functions  $f = x^a$  and  $g = x^b$  it leads to the star-product commutator  $[x^a, x^b]_\star = i\theta^{ab}$ , the defining relation of the Moyal-Weyl plane.
2. The star-product is associative  $f \star (g \star h) = (f \star g) \star h$ . This is consistent with the associativity of the operator product expansion in conformal field theory. Actually, in CFT one requires crossing symmetry of correlation functions, which implies the weaker constraint of associativity up to boundary term, i.e.  $\int dx f \star (g \star h) - \int dx (f \star g) \star h = 0$ .
3. It is also cyclic in the sense  $\int dx f \star g = \int dx g \star f$  and similarly for cyclic permutations of the product of  $N$  functions. This is a consequence of the fact that the conformal  $SL(2, \mathbb{R})$  symmetry group leaves the cyclic order of the inserted vertex operators invariant.

So far we considered the most tractable case of a constant background flux in flat space, but mathematically the star-product was defined in [19] for an arbitrary Poisson structure  $\theta^{ij}$ . In this case the corresponding star-product is still associative. However one can also consider the same product for only a quasi Poisson structure, which then leads to a nonassociative star-product (see e.g. [8]). Physically, also in this most generic situation, as long as the background satisfies the string equations of motion, i.e. for an on-shell background, the properties 1.-3. above should still be satisfied. Following essentially [13, 20, 21], let us confirm this for the first non-trivial order terms.

Now we have the physical situation of an open string ending on a D-brane with generic non-constant  $B$ -field, i.e. non-vanishing field strength  $H$  supported on a brane embedded in some generically curved space. At leading order in derivatives this leads to a noncommutative product

$$\begin{aligned} f \circ g = f \cdot g + i \frac{1}{2} \theta^{ab} \partial_a f \partial_b g - \frac{1}{8} \theta^{ab} \theta^{cd} \partial_a \partial_c f \partial_b \partial_d g \\ - \frac{1}{12} (\theta^{am} \partial_m \theta^{bc}) (\partial_a \partial_b f \partial_c g + \partial_a \partial_b g \partial_c f) \dots \end{aligned} \quad (33)$$

where the first line is just the expansion of the Moyal-Weyl product (32). The associator for this product becomes

$$(f \circ g) \circ h - f \circ (g \circ h) = \frac{1}{6} \theta^{abc} \partial_a f \partial_b g \partial_c h + O(\partial \theta^2) \quad (34)$$

with  $\theta^{abc} = 3 \theta^{[am} \partial_m \theta^{bc]}$ , which by definition vanishes for a Poisson tensor. This can be written as

$$\theta^{abc} = \theta^{am} \theta^{bn} \theta^{cp} (2\pi\alpha' dF + H)_{mnp} = \theta^{am} \theta^{bn} \theta^{cp} H_{mnp} \quad (35)$$

where we used the Bianchi identity  $dF = 0$ .

For a slowly varying gauge field, the effective action is given by the Dirac-Born-Infeld (DBI) action

$$S_{\text{DBI}} = \int dx \sqrt{g + 2\pi\alpha' \mathcal{F}}. \quad (36)$$

Varying it with respect to the gauge potential  $A$  in  $2\pi\alpha' \mathcal{F} = B + 2\pi\alpha' dA$ , one gets the equation of motion

$$\partial_a \left( \sqrt{g + 2\pi\alpha' \mathcal{F}} [(g + 2\pi\alpha' \mathcal{F})^{-1}]^{[ab]} \right) = \frac{1}{2\pi\alpha'} \partial_a \left( \sqrt{g + 2\pi\alpha' \mathcal{F}} \theta^{ab} \right) = 0 \quad (37)$$

where we have used (30). Then, it directly follows that up to leading order in  $\partial\theta$  the  $\circ$ -product satisfies cyclicity

$$\int dx \sqrt{g + 2\pi\alpha' \mathcal{F}} f \circ g = \int dx \sqrt{g + 2\pi\alpha' \mathcal{F}} g \circ f. \quad (38)$$

Indeed, e.g. at order  $O(\theta)$  the difference between the left and the right hand side is a total derivative on-shell

$$\frac{i}{2} \int dx \sqrt{g + 2\pi\alpha' \mathcal{F}} \theta^{ab} \partial_a f \partial_b g = \frac{i}{2} \int dx \partial_a \left( \sqrt{g + 2\pi\alpha' \mathcal{F}} \theta^{ab} f \partial_b g \right) = 0 \quad (39)$$

Thus, as expected from CFT, the product of two functions is cyclic, once the background satisfies the string equations of motion.

Similarly, the associator below the integral also gives a total derivative at leading order in  $\partial\theta$

$$\begin{aligned} \int dx \sqrt{g + 2\pi\alpha' \mathcal{F}} \left( (f \circ g) \circ h - f \circ (g \circ h) \right) \\ = \frac{1}{6} \int dx \sqrt{g + 2\pi\alpha' \mathcal{F}} \left( \theta^{am} \theta^{bn} \theta^{cp} H_{mnp} \partial_a f \partial_b g \partial_c h + O(\partial \theta^2) \right) \\ = \frac{1}{6} \int dx \partial_a \left( \sqrt{g + 2\pi\alpha' \mathcal{F}} \theta^{abc} f \partial_b g \partial_c h \right) + O(\partial \theta^2). \end{aligned} \quad (40)$$

Thus, we conclude that, as expected from the open string conformal field theory, on-shell (up to  $O(\partial \theta^2)$ ) the  $\circ$ -product is associative up to boundary terms. Our argument seems to suggest that it might be relevant for string field theory in off-shell backgrounds, but to our knowledge this has not been made explicit.

#### 4 Noncommutative geometry: closed string generalization

In the previous section we have seen that noncommutativity arises for open strings in a magnetic flux background leading to noncommutative gauge theories. Clearly, if noncommutative geometry has any significance for improving the ultraviolet behavior of quantum gravity, it should also appear for closed strings. Thinking about this question, one realizes that the closed string analogue must be different as here two vertex operators are inserted in the bulk of a two-sphere  $S^2$  and no unambiguous ordering of  $z_1$  and  $z_2$  can be defined. Therefore, one does not expect the same kind of noncommutativity to arise.



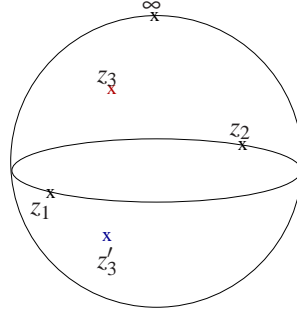
However, as shown in figure 3, adding one more point might improve the situation. Just looking at the  $S^2$  geometrically,  $z_1$  and  $z_2$  define a geodesic. Then adding  $z_3$  one can distinguish the two situation that it lies on the same or opposite hemisphere as the point  $\infty$ . Then following the open string logic, turning on a three-vector flux  $\Theta^{abc}$  distinguishing these two configurations can make this tri-product structure manifest. Thus, in this case one would e.g. expect a fundamental non-vanishing tri-product of the form<sup>1</sup>

$$(f_1 \triangle f_2 \triangle f_3)(x) = \exp\left(\Theta^{abc} \partial_a^{x_1} \partial_b^{x_2} \partial_c^{x_3}\right) f_1(x_1) f_2(x_2) f_3(x_3) \Big|_x \quad (41)$$

which allows to define a totally antisymmetric tri-bracket

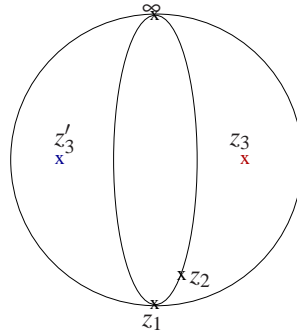
$$[f_i, f_j, f_k] = \sum_{\sigma \in S_3} \text{sign}(\sigma) f_{\sigma(i)} \triangle f_{\sigma(j)} \triangle f_{\sigma(k)}, \quad (42)$$

where  $S_3$  denotes the permutation group of three elements.



**Fig. 3** Geometric four punctured sphere

However, there is a caveat to this simple picture, namely that in string theory one has to take into account the conformal  $SL(2, \mathbb{C})$  symmetry of the world-sheet. As shown in figure 4, this allows to map the three points  $z_1$ ,  $z_2$  and the north pole to  $z_1 = 0$ ,  $z_2 = 1$  and  $z_4 = \infty$ , i.e. they do lie on the same geodesic. Therefore, adding  $z_3$  does not lead to any clearly distinguished order. This is consistent with the fact that in CFT a four-point function can be expressed in terms of the  $SL(2, \mathbb{C})$  invariant cross-ratio and features crossing symmetry. The latter is nothing else than associativity of the operator product expansion.



**Fig. 4** CFT sphere diagram

From these simple arguments we arrive at the two features for the generalization of noncommutativity to the closed string sector:

<sup>1</sup> One could also try to not consider this tri-product to be fundamental but the result of a non-vanishing Jacobiator for a generalized star-product. This approach was initiated in [5] and is discussed in the lecture by P. Schupp (see the lecture notes [22] and also [8, 10–12]).

1. Noncommutativity/nonassociativity is expected to rather involve a three-vector  $\Theta^{abc}$  and a corresponding tri-product.
2. However, its effect is expected to be invisible in the CFT.

Clearly, with these insights one could stop at this stage. However, since these ideas have the potential to be a window into off-shell string theory or at least into structures present in theories beyond the usual approach to string theory, let us proceed and investigate how precisely the two features 1. and 2. above are realized in closed string theory.

First we will clarify what the nature of the three-vector flux  $\Theta^{abc}$  might be. In fact in the study of nongeometric fluxes, precisely such a candidate appeared, the so-called  $R$ -flux. Since the appropriate framework for describing it is double field theory we will also present some of its structure. For a more detailed presentation we refer the reader to the review articles [23–25].

Second, we will perform a CFT analysis for a constant flux background and, in analogy to the open string story, will compute correlation functions of vertex operators.

### The nature of the flux $\Theta^{abc}$

In closed string theory there generically exists a massless Kalb-Ramond two form, whose field strength  $H = dB$  is a three-form. At first sight, this seems to be the natural candidate for  $\Theta^{abc}$ . Whereas a constant two-form flux on a D-brane only slightly changes the boundary conditions of the open string, due to (9), a constant  $H$ -flux really backreacts on the geometry and also implies that the underlying exact CFT is not any longer free but becomes highly interacting. Therefore, this situation is much harder to analyze.

However, the  $H$ -flux is not the only candidate for a source of nonassociativity. Applying T-duality to the closed string background [26] given by a flat space with constant non-vanishing three-form flux  $H = dB$ , results in a background with geometric flux. This so-called twisted torus is still a conventional string background, but a second T-duality leads to a nongeometric flux background. These are spaces in which the transition functions between two charts of a manifold are allowed to be T-duality transformations, hence they are also called T-folds [27, 28]. After formally applying a third T-duality, not along an isometry direction anymore, one obtains an  $R$ -flux background which does not admit a clear target-space interpretation. It was proposed that this background does not correspond to an ordinary geometry even locally, but instead gives rise to a nonassociative geometry [29]. Therefore, this nongeometric flux seems to be the prime candidate for  $\Theta^{abc}$ .

As mentioned, for the last T-duality giving  $R$ -flux one cannot apply the Buscher rules. During the last years a formalism was developed which allows to precisely perform such T-dualities. This is called double field theory (DFT) and is a field theory for the massless modes of the bosonic string which has manifest  $O(D, D)$  symmetry. A full introduction into this fast developing subject is beyond the scope of this lecture and we refer to [23–25] for more details. Here we just give a very few ingredients which allow to perform the series of T-dualities mentioned above.

The main idea is that one doubles the coordinates  $X^A = (x^a, \tilde{x}_a)$  by introducing so-called winding coordinates  $\tilde{x}_a$ . This allows to put the diffeomorphism symmetry and the  $B$ -field gauge transformations on equal footing by combining the dynamical fields  $g_{ab}$  and  $B_{ab}$  in a generalized metric

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{ab} & -g^{am}B_{mb} \\ B_{am}g^{mb} & g_{ab} - B_{am}g^{mn}B_{nb} \end{pmatrix}. \quad (43)$$

The global  $h \in O(D, D)$  symmetry acts as

$$\mathcal{H}' = h' \mathcal{H} h, \quad X' = h X, \quad \partial' = (h')^{-1} \partial. \quad (44)$$

One can construct an  $O(D, D)$  invariant action for this generalized metric, where eventually one halves the degrees of freedom by imposing the so-called strong constraint

$$\partial_a \partial^a = 0, \quad \partial_a f \tilde{\partial}^a g + \tilde{\partial}_a f \partial^a g = 0 \quad (45)$$

where  $f, g$  denote the fundamental objects in DFT like  $\mathcal{H}_{AB}$  and gauge parameters.

In order to see how T-duality is described in this framework consider a flat torus  $T^3$  with a constant  $H$ -flux with  $B_{12} = hx^3$ . The generalized metric then takes the form<sup>2</sup>

$$\mathcal{H} = \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -hx^3 & 0 \\ 0 & 1 & 0 & hx^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & hx^3 & 0 & 1+(hx^3)^2 & 0 & 0 \\ -hx^3 & 0 & 0 & 0 & 1+(hx^3)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \quad (46)$$

A T-duality in the  $x_1$  direction is a special  $O(D, D)$  transformation and acts via conjugation  $\mathcal{H}' = \mathcal{T}_1^t \mathcal{H} \mathcal{T}_1$  with the matrix

$$\mathcal{T}_1 = \left( \begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \quad (47)$$

One gets

$$\mathcal{H}' = \left( \begin{array}{ccc|ccc} 1+(hx^3)^2 & hx^3 & 0 & 0 & 0 & 0 \\ hx^3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -hx^3 & 0 \\ 0 & 0 & 0 & -hx^3 & 1+(hx^3)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad (48)$$

which corresponds to the metric

$$ds^2 = (dx^1 - hx^3 dx^2)^2 + (dx^2)^2 + (dx^3)^2 \quad (49)$$

on the twisted three torus with vanishing  $B$ -field. Thus, this background is characterized by a constant geometric flux  $F^1_{23} = h$ .

Second, let us perform a T-duality in the isometric  $x_2$  direction. The generalized metric transforms as  $\mathcal{H}'' = \mathcal{T}_2^t \mathcal{H}' \mathcal{T}_2$  with

$$\mathcal{H}'' = \left( \begin{array}{ccc|ccc} 1+(hx^3)^2 & 0 & 0 & 0 & hx^3 & 0 \\ 0 & 1+(hx^3)^2 & 0 & -hx^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -hx^3 & 0 & 1 & 0 & 0 \\ hx^3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \quad (50)$$

Reading off the metric and the B-field, we find

$$g = \left( \begin{array}{ccc} \frac{1}{1+(hx^3)^2} & 0 & 0 \\ 0 & \frac{1}{1+(hx^3)^2} & 0 \\ 0 & 0 & 1 \end{array} \right), \quad B = \left( \begin{array}{ccc} 0 & -\frac{hx^3}{1+(hx^3)^2} & 0 \\ \frac{hx^3}{1+(hx^3)^2} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \quad (51)$$

<sup>2</sup> Except for the last step, the  $R$ -flux, the following computation can be found in the appendix of [30].

Due to the terms in the denominators, this looks a bit awkward. However, in generalized geometry and DFT one can also parametrize the generalized metric via

$$\mathcal{H}_{ab} = \begin{pmatrix} \tilde{g}^{ab} - \beta^{am} \tilde{g}_{mn} \beta^{nb} & \beta^{am} \tilde{g}_{mb} \\ -\tilde{g}_{am} \beta^{mb} & \tilde{g}_{ab} \end{pmatrix} \quad (52)$$

where  $\beta = \frac{1}{2} \beta^{ab} \partial_a \wedge \partial_b$  denotes an anti-symmetric bi-vector. The geometric frame (43) and this nongeometric one are related via the field redefinition

$$\begin{aligned} \tilde{g} &= g - B g^{-1} B, \\ \beta &= -\tilde{g}^{-1} B g^{-1}, \end{aligned} \quad (53)$$

which is reminiscent of the Buscher rules and the open string relation (30). Using this frame one gets the simple result

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & h x^3 & 0 \\ -h x^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (54)$$

Then one defines the nongeometric  $Q$  flux as

$$Q_3^{12} = \partial_3 \beta^{12} = h \quad (55)$$

so that this background has a flat metric  $\tilde{g}$  and a constant  $Q$ -flux.

Thus, we have established a chain of T-duality transformations

$$H_{ijk} \leftrightarrow F_{ij}{}^k \leftrightarrow Q_i{}^{jk} \leftrightarrow R^{ijk} \quad (56)$$

where we have indicated the final step, namely the result of performing a T-duality also in the non-isometric  $x_3$  direction. We note that, according to the  $O(D, D)$  action (44) for  $\mathcal{T}_3$ , the normal and winding coordinate are exchanged as  $x_3 \leftrightarrow \tilde{x}_3$  so that

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & h \tilde{x}_3 & 0 \\ -h \tilde{x}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (57)$$

Then one defines the nongeometric  $R$  flux as

$$R^{123} = 3 \tilde{\partial}^{[3} \beta^{12]} = h. \quad (58)$$

Therefore, this flux explicitly involves winding derivatives and is not visible in ordinary geometry.

### CFT analysis of $\Theta^{abc}$ flux

Now, we will analyze the conformal field theory for a closed string moving on a flat space with a constant  $H$ -flux background. The aim is to perform a similar computation as for the open string, i.e. in particular to compute  $N$ -point functions of tachyon vertex operators. But before we can do that, we need to define a CFT. The essential observation is that a Ricci flat metric, vanishing dilaton and a constant  $H$ -flux solves the string equation of motion up to linear order in  $H$ . Since we are interested in a linear order effect, it is appropriate to perform conformal perturbation theory up to linear order in  $H$  around the trivial background. Therefore, the starting point is a flat metric and a constant  $H$ -flux specified by

$$ds^2 = \sum_{a=1}^N (dX^a)^2, \quad H = \frac{2}{\alpha'^2} \Theta_{abc} dX^a \wedge dX^b \wedge dX^c, \quad (59)$$

where for simplicity we focus on  $N = 3$ . As said, the expectation is that this background corresponds to a CFT up to linear order in  $H$ . Moreover, since the origin of T-duality lies in conformal field theory where it is nothing else than an asymmetric reflection  $(X_L, X_R) \rightarrow (X_L, -X_R)$ , one can try to analyze the  $R$ -flux case also from the CFT point of view. The remainder of this article is essentially a brief version of the more exhaustive analysis originally presented in [6] and reviewed in [14].

To proceed, we write the action (1) as the sum of a free part  $\mathcal{S}_0$  and a perturbation  $\mathcal{S}_1$ . Choosing a gauge such that  $B_{ab} = \frac{1}{3}H_{abc}X^c$ , we have

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 \quad \text{with} \quad \mathcal{S}_1 = \frac{1}{2\pi\alpha'} \frac{H_{abc}}{3} \int_{\Sigma} d^2z X^a \partial X^b \bar{\partial} X^c. \quad (60)$$

We expect  $\mathcal{S}_1$  to be a marginal operator (only) up to linear order in  $H$ . A correlation function of  $N$  operators  $\mathcal{O}_i[X]$  can be computed via the path integral in the usual way

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \frac{1}{\mathcal{Z}} \int [dX] \mathcal{O}_1 \dots \mathcal{O}_N e^{-\mathcal{S}[X]}, \quad (61)$$

where  $\mathcal{Z}$  denotes the vacuum functional given by  $\mathcal{Z} = \int [dX] e^{-\mathcal{S}[X]}$ . In the limit of small fluxes, we can do conformal perturbation theory and expand (61) in the perturbation  $\mathcal{S}_1$  leading to

$$\begin{aligned} \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle &= \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_0 - \langle \mathcal{O}_1 \dots \mathcal{O}_N \mathcal{S}_1 \rangle_0 \\ &+ \frac{1}{2} \left[ \langle \mathcal{O}_1 \dots \mathcal{O}_N \mathcal{S}_1^2 \rangle_0 - \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_0 \times \langle \mathcal{S}_1^2 \rangle_0 \right] + \mathcal{O}(H^3). \end{aligned} \quad (62)$$

First, we compute the correction to the three-point functions of three currents  $J^a = i\partial X^a$ ,  $\bar{J}^a = i\bar{\partial} X^a$ . It turns out that there are also non-vanishing correlators like  $\langle J^a J^b \bar{J}^c \rangle$ , i.e. the currents are not holomorphic respectively anti-holomorphic. However, one can define new fields  $\mathcal{J}^a$  and  $\bar{\mathcal{J}}^a$

$$\begin{aligned} \mathcal{J}^a(z, \bar{z}) &= J^a(z) - \frac{1}{2} H^a_{bc} J^b(z) X_R^c(\bar{z}), \\ \bar{\mathcal{J}}^a(z, \bar{z}) &= \bar{J}^a(\bar{z}) - \frac{1}{2} H^a_{bc} X_L^b(z) \bar{J}^c(\bar{z}) \end{aligned} \quad (63)$$

so that the three current correlators take the CFT form

$$\begin{aligned} \langle \mathcal{J}^a(z_1, \bar{z}_1) \mathcal{J}^b(z_2, \bar{z}_2) \mathcal{J}^c(z_3, \bar{z}_3) \rangle &= -i \frac{\alpha'^2}{8} H^{abc} \frac{1}{z_{12} z_{23} z_{13}}, \\ \langle \bar{\mathcal{J}}^a(z_1, \bar{z}_1) \bar{\mathcal{J}}^b(z_2, \bar{z}_2) \bar{\mathcal{J}}^c(z_3, \bar{z}_3) \rangle &= +i \frac{\alpha'^2}{8} H^{abc} \frac{1}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{13}}. \end{aligned} \quad (64)$$

The necessity of this redefinition can already be understood from the two-dimensional equation of motion  $\partial \bar{\partial} X^a = \frac{1}{2} H^a_{bc} \partial X^b \bar{\partial} X^c$ . Therefore, already at linear order the coordinate fields have to be adjusted to be consistent with a CFT description. However, the deformation is still marginal, which means that there are no infinities leading to a renormalization group flow.

Writing the new currents as derivatives of corrected coordinates  $\mathcal{X}^a$ , after three integrations the three-point function of these coordinates can be computed as

$$\langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle^H = \Theta^{abc} \left[ \mathcal{L} \left( \frac{z_{12}}{z_{13}} \right) - \mathcal{L} \left( \frac{\bar{z}_{12}}{\bar{z}_{13}} \right) \right] \quad (65)$$

with  $\Theta^{abc} = \frac{\alpha'^2}{12} H^{abc}$  and

$$\mathcal{L}(z) = L(z) + L\left(1 - \frac{1}{z}\right) + L\left(\frac{1}{1-z}\right) \quad \text{where} \quad L(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1-z) \quad (66)$$

denotes the (complex) Rogers dilogarithm<sup>3</sup>. It satisfies the so-called fundamental identity  $L(z) + L(1-z) = L(1)$ . The three-point function (65) should be considered as the closed string generalization of the second term in (28). However, there are two essential differences:

- For the closed string it is the three- and not the two-point function which is corrected.
- For the closed string the Rogers dilogarithm gives rise to a non-trivial world-sheet dependence, whereas for the open string only the essentially constant step-function appeared.

One can also compute the correction to the two-point function of two coordinates. It reads

$$\delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle = \frac{\alpha'^2}{8} H^a{}_{pq} H^{bpq} \log |z_1 - z_2|^2 \log \varepsilon, \quad (67)$$

where  $\varepsilon$  is a cut-off. Therefore, we explicitly see that the perturbation  $\mathcal{S}_1$  ceases to be marginal at second order in the flux. The theory is no longer conformally invariant and starts to run according to the renormalization group flow equation for the inverse world-sheet metric  $G^{ab}$ , which is of the form

$$\mu \frac{\partial G^{ab}}{\partial \mu} = -\frac{\alpha'}{4} H^a{}_{pq} H^{bpq}. \quad (68)$$

This precisely agrees with equation (9) for constant space-time metric,  $H$ -flux and dilaton.

Even though in the framework of the Buscher rules, applying three T-dualities on the  $H$ -flux background is questionable, on the level of the CFT, T-duality corresponds to a simple asymmetric transformation of the world-sheet theory. It is just a reflection of the right-moving coordinates. Since our corrected fields  $\mathcal{X}^a(z, \bar{z})$  still admit a split into a holomorphic and an anti-holomorphic piece, we define T-duality on the world-sheet action along the direction  $\mathcal{X}^a$  as

$$\begin{array}{ccc} \mathcal{X}_L^a(z) & \xrightarrow{\text{T-duality}} & +\mathcal{X}_L^a(z), \\ \mathcal{X}_R^a(\bar{z}) & & -\mathcal{X}_R^a(\bar{z}). \end{array} \quad (69)$$

Under a T-duality in all three directions, momentum modes in the  $H$ -flux background are mapped to winding modes in the  $R$ -flux background. We are now interested in momentum modes in the  $R$ -flux background which are related via T-duality to winding modes in the  $H$ -flux background. Therefore, the three-point function in the  $R$ -flux background should read

$$\langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle^R = \Theta^{abc} \left[ \mathcal{L} \left( \frac{z_{12}}{z_{13}} \right) + \mathcal{L} \left( \frac{\bar{z}_{12}}{\bar{z}_{13}} \right) \right], \quad (70)$$

which just has a different relative sign between the holomorphic and anti-holomorphic part. Here, we have the relation  $\Theta^{abc} = \frac{\alpha'^2}{12} R^{abc}$ .

Up to linear order in the flux we can write the energy-momentum tensor as

$$\mathcal{T}(z) = \frac{1}{\alpha'} \delta_{ab} : \mathcal{J}^a \mathcal{J}^b : (z), \quad \overline{\mathcal{T}}(\bar{z}) = \frac{1}{\alpha'} \delta_{ab} : \overline{\mathcal{J}}^a \overline{\mathcal{J}}^b : (\bar{z}). \quad (71)$$

They give rise to two copies of the Virasoro algebra with central charge  $c = 3$  and  $\mathcal{J}^a(\overline{\mathcal{J}}^a)$  is indeed a (anti-)chiral primary field with  $h = 1$  ( $\bar{h} = 1$ ). Recall that the aim is to carry out a similar computation as for the open string case, i.e. to evaluate string scattering amplitudes for vertex operators and to see whether there is any sign of a new space-time noncommutative/nonassociative product. In the free theory the tachyon vertex operator is a primary field of conformal dimension  $(h, \bar{h}) = (\frac{\alpha'}{4} p^2, \frac{\alpha'}{4} p^2)$ , and in covariant

<sup>3</sup> The theory of the complex Rogers dilogarithm is more involved than the real analogue. Some its features are collected in the appendix of [6].

quantization of the bosonic string physical states are given by primary fields of conformal dimension  $(h, \bar{h}) = (1, 1)$ . The natural definition of the tachyon vertex operator for the perturbed theory is

$$\mathcal{V}(z, \bar{z}) = : \exp(ip \cdot (\mathcal{X}_L + \mathcal{X}_R)) : . \quad (72)$$

One can compute

$$\mathcal{T}(z_1) \mathcal{V}(z_2, \bar{z}_2) = \frac{1}{(z_1 - z_2)^2} \frac{\alpha' p \cdot p}{4} \mathcal{V}(z_2, \bar{z}_2) + \frac{1}{z_1 - z_2} \partial \mathcal{V}(z_2, \bar{z}_2) + \text{reg.} , \quad (73)$$

and analogously for the anti-holomorphic part. This means that the vertex operator (72) is primary and has conformal dimension  $(h, \bar{h}) = (\frac{\alpha'}{4} p^2, \frac{\alpha'}{4} p^2) = (1, 1)$ . It is therefore a physical quantum state of the deformed theory. For the correlator of three tachyon vertex operators one obtains

$$\langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle^{H/R} = \frac{\delta(p_1 + p_2 + p_3)}{|z_{12} z_{13} z_{23}|^2} \exp \left[ -i \Theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[ \mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right] \right]_{\Theta} , \quad (74)$$

where  $[\dots]_{\Theta}$  indicates that the result is valid only up to linear order in  $\Theta$ . The full string scattering amplitude of the integrated tachyon vertex operators then becomes

$$\begin{aligned} \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \rangle^{H/R} &= \int \prod_{i=1}^3 d^2 z_i \delta^{(2)}(z_i - z_i^0) \delta(p_1 + p_2 + p_3) \times \\ &\exp \left[ -i \Theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[ \mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right] \right]_{\Theta} . \end{aligned} \quad (75)$$

Let us now study the behavior of (74) under permutations of the vertex operators. Before applying momentum conservation, the three-tachyon amplitude for a permutation  $\sigma \in S_3$  of the vertex operators can be computed using the relation  $L(z) + L(1-z) = L(1)$ . With  $\varepsilon = -1$  for the  $H$ -flux and  $\varepsilon = +1$  for the  $R$ -flux, one finds

$$\langle \mathcal{V}_{\sigma(1)} \mathcal{V}_{\sigma(2)} \mathcal{V}_{\sigma(3)} \rangle^{H/R} = \exp \left[ i \left( \frac{1+\varepsilon}{2} \right) \eta_{\sigma} \pi^2 \Theta^{abc} p_{1,a} p_{2,b} p_{3,c} \right] \langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle^{H/R} , \quad (76)$$

where in addition  $\eta_{\sigma} = 1$  for an odd permutation and  $\eta_{\sigma} = 0$  for an even one. One observes that for  $H$ -flux the phase is always trivial while for  $R$ -flux a non-trivial phase may appear. Recall that our analysis is only reliable up to linear order in  $\Theta^{abc}$ .

Note that it is non-trivial that this phase is independent of the world-sheet coordinates, which can be traced back to the form of the fundamental identity of  $L(z)$ . For this reason, it can be thought of as a property of the underlying target space. Indeed, the phase in (76) can be recovered from a new three-product on the space of functions  $V_{p_n}(x) = \exp(ip_n \cdot x)$  which is defined as

$$V_{p_1}(x) \triangle V_{p_2}(x) \triangle V_{p_3}(x) \stackrel{\text{def}}{=} \exp \left( -i \frac{\pi^2}{2} \Theta^{abc} p_{1,a} p_{2,b} p_{3,c} \right) V_{p_1+p_2+p_3}(x) . \quad (77)$$

However, in CFT correlation functions operators are understood to be radially ordered and so changing the order of operators should not change the form of the amplitude. This is known as crossing symmetry and is also called the associativity of the operator product expansion. In the case of the  $R$ -flux background, this is reconciled by applying momentum conservation leading to

$$p_{1,a} p_{2,b} p_{3,c} \Theta^{abc} = 0 \quad \text{for} \quad p_3 = -p_1 - p_2 . \quad (78)$$

Therefore, scattering amplitudes of three tachyons do not receive any corrections at linear order in  $\Theta$  both for the  $H$ - and  $R$ -flux.

The tri-product (77) can be generalized to more generic functions as

$$f_1(x) \triangle f_2(x) \triangle f_3(x) \stackrel{\text{def}}{=} \exp \left( \frac{\pi^2}{2} \Theta^{abc} \partial_a^{x_1} \partial_b^{x_2} \partial_c^{x_3} \right) f_1(x_1) f_2(x_2) f_3(x_3) \Big|_x , \quad (79)$$

where we used the notation  $(\ )|_x = (\ )|_{x_1=x_2=x_3=x}$ . This is to be compared with the  $\star$ -product (32) and can be thought of as a possible closed string generalization of the open string noncommutative structure. For the tri-bracket of the coordinates  $x^a$  one gets  $[x^a, x^b, x^c] = 3\pi^2 \Theta^{abc}$ .

This result generalizes to the  $N$ -tachyon amplitude, where the relative phase can be described by the following deformed product

$$f_1(x) \triangle_N \dots \triangle_N f_N(x) \stackrel{\text{def}}{=} \exp \left[ \frac{\pi^2}{2} \Theta^{abc} \sum_{1 \leq i < j < k \leq N} \partial_a^{x_i} \partial_b^{x_j} \partial_c^{x_k} \right] f_1(x_1) f_2(x_2) \dots f_N(x_N) \Big|_x, \quad (80)$$

which has the peculiar feature that the phase becomes trivial after taking momentum conservation into account or equivalently

$$\int d^n x f_1(x) \triangle_N f_2(x) \triangle_N \dots \triangle_N f_N(x) = \int d^n x f_1(x) f_2(x) \dots f_N(x). \quad (81)$$

We conclude that for constant closed string  $R$ -flux, a non-vanishing tri-product (79) and (80) is consistent with the main axioms of a conformal field theory. Below the integral it only gives boundary terms and therefore does not change the equations of motion. Therefore, we have identified a non-trivial nonassociative deformation of the underlying target space, which is not visible in CFT. Whether this deformation is “real” can only be decided, if one goes beyond conformal field theory, i.e. the usual perturbative approach to string theory. Some simple ideas in this direction will be discussed in the following final part of this lecture.

### Beyond constant $\Theta^{abc}$ flux

When the lecture was given, the question arose what happens for non-constant  $\Theta$ -flux and whether it helps to decide about the relevance of this entire approach. Thus, the question is how the open string analysis from the end of section 3 generalizes to the closed string. Since nongeometric fluxes are appropriately described in double field theory (DFT), the analysis should be performed in this framework. This was indeed done in [13]. For self-consistency of this lecture, here we will not work in the most general DFT setting but instead present some of the essential results in a simplified form.

For the open string, it happened that both cyclicity and associativity hold up to boundary terms precisely if the string equations of motion were satisfied for the background. Let us consider the tri-product (79) of three functions below an integral on a manifold with curved metric  $g_{ab}$

$$\int dx \sqrt{-g} e^{-2\phi} f_1 \triangle f_2 \triangle f_3 = \int dx \sqrt{-g} e^{-2\phi} f_1 f_2 f_3 + \frac{\pi^2}{2} \int dx \sqrt{-g} e^{-2\phi} \Theta^{abc} \partial_a f_1 \partial_b f_2 \partial_c f_3 + \dots \quad (82)$$

Performing an integration by parts for the second term, we get

$$\begin{aligned} \int dx \sqrt{-g} e^{-2\phi} \Theta^{abc} \partial_a f_1 \partial_b f_2 \partial_c f_3 &= \int dx \partial_a \left( \sqrt{-g} e^{-2\phi} \Theta^{abc} f_1 \partial_b f_2 \partial_c f_3 \right) \\ &\quad - \int dx \partial_a \left( \sqrt{-g} e^{-2\phi} \Theta^{abc} \right) f_1 \partial_b f_2 \partial_c f_3 \end{aligned} \quad (83)$$

which gives a boundary term for

$$\partial_a \left( \sqrt{-g} e^{-2\phi} \Theta^{abc} \right) = 0. \quad (84)$$

This is precisely the leading order equation of motion for the  $H$ -flux, i.e. for  $\Theta^{abc} = g^{aa'} g^{bb'} g^{cc'} H_{a'b'c'}$ . Therefore, for such a tri-product, its effect at linear order (nonassociativity described by a tri-bracket)



gives a boundary term. In [13] it was shown that this behavior also holds for a doubled flux  $F^{ABC}$  which includes also the gravitational part.

However, this is actually *not* the flux, for which the tri-product appeared in the CFT. There, we argued that it is the  $R$ -flux, i.e.  $\Theta^{abc} = R^{abc}$ , which by itself is already an anti-symmetric three-vector so that no metric factors are involved. Therefore, in this sense such a tri-product is rather of topological type and should involve Bianchi identities instead of equations of motion. In [13] this was shown to be essentially correct in the DFT framework. However, in DFT the  $O(D, D)$  symmetry requires that with  $R^{abc} = 3 \partial^{[a} \beta^{bc]}$  the complete tri-product takes the form

$$f_1 \triangle f_2 \triangle f_3 = f_1 f_2 f_3 + \frac{\pi^2}{2} \left( R^{abc} \partial_a f_1 \partial_b f_2 \partial_c f_3 + Q_a{}^{bc} (\tilde{\partial}^a f_1 \partial_b f_2 \partial_c f_3 + \text{cycl}_{f_1, f_2, f_3}) \right). \quad (85)$$

Employing the strong constraint (45) between  $\beta$  and the  $f_i$ , the entire second term on the right hand side vanishes and the tri-product becomes trivial. Thus, in DFT one can only get a non-trivial tri-product if the strong constraint is weakened. A more elaborate discussion can be found in [13].

This reflects the status of the emergence of nonassociativity in closed string theory and it is fair to say that this issue is still not completely settled and more work is needed to put together the bits and pieces that were revealed so far.

**Acknowledgements** I would like to thank my collaborators Andre Betz, Andreas Deser, Michael Fuchs, Xin Gao, Falk Hassler, Daniela Herschmann, Dieter Lüst, Erik Plauschinn, Felix Rennecke, Christian Schmid, Pramod Shukla and Rui Sun for the many discussions on the subject presented in this mini course. Moreover, I thank the organizers of the Corfu Workshop on Noncommutative Field Theory and Gravity September 8 - 15, 2013, for creating such a stimulating atmosphere and the participants for asking so many good questions.

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